Discrete Mathematics 71 (1988) 119–127
North-Holland

RAMSEY NUMBERS BASED ON C_5-DECOMPOSITIONS

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Received 4 September 1986
Received 17 August 1987

Multicolor Ramsey numbers are given for some small, dense graphs. The lower bounds are established using constructions which can be nearly decomposed into vertex-disjoint copies of C_5.

Introduction

This paper reports on an effort to determine some multicolor Ramsey numbers for small, dense graphs. Progress on this problem has been slow. Our original, perhaps excessively ambitious, goal was to complete a table of three color numbers similar to the tables for two colors presented by Chvátal and Harary in [3, 4] and by Clancy in [5].

In the process, we observed that several of the constructions which have been used to establish known Ramsey numbers have similarities of structure. Specifically, their adjacency matrices contain square submatrices of order 5 which are themselves adjacency matrices of C_5. Below are four examples of such constructions. Throughout, X will denote the 5×5 circulant matrix with first row $(0, 1, 0, 0, 1)$ and Y will denote the circulant matrix with first row $(0, 0, 1, 1, 0)$. The all-ones matrix is denoted by J. Observe that $J = X + Y + I$.

Example 1. The Ramsey numbers $r(K_3) = 6$ and $r(C_4) = 6$ are established by decomposing K_5 into two copies of C_5, one with adjacency matrix X, the other with adjacency matrix Y [3].

Example 2. The graph used to determine the Ramsey number of K_5-e in [6] has

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21 vertices and its adjacency matrix is

$$\begin{bmatrix} X & X & I & J-I & \begin{matrix}1\\1\\1\\1\\1\end{matrix} \\ X & X & J-I & I & \begin{matrix}1\\1\\1\\1\\1\end{matrix} \\ I & J-I & Y & Y+1 & \begin{matrix}0\\0\\0\\0\\0\end{matrix} \\ J-I & I & Y+I & Y & \begin{matrix}0\\0\\0\\0\\0\end{matrix} \\ 11111 & 11111 & 00000 & 00000 & 0 \end{bmatrix}$$

Example 3. As indicated in [6], $r(C_4, K_5\text{-}e) = 11$. One way to prove the lower bound is to use the Petersen graph. And the adjacency matrix for the Petersen graph can be expressed as:

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix}.$$

Example 4. In [9], Greenwood and Gleason established the only known exact multicolor Ramsey number for complete graphs. The coloring of K_{16} which they used to show that $r(K_3, K_3, K_3) = 17$ can be described by adjacency matrices M_1, M_2 and M_3 for the three colors as follows:

$$M_1 = \begin{bmatrix} X & I & X & \begin{matrix}0\\0\\0\\0\\0\end{matrix} \\ I & Y & Y & \begin{matrix}0\\0\\0\\0\\0\end{matrix} \\ X & Y & 0 & \begin{matrix}1\\1\\1\\1\\1\end{matrix} \\ 00000 & 00000 & 11111 & 0 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 0 & X & Y & \begin{matrix}1\\1\\1\\1\\1\end{matrix} \\ X & X & I & \begin{matrix}0\\0\\0\\0\\0\end{matrix} \\ Y & I & Y & \begin{matrix}0\\0\\0\\0\\0\end{matrix} \\ 11111 & 00000 & 00000 & 0 \end{bmatrix}$$

$$M_3=\left[\begin{array}{cccc} Y & Y & I & \begin{smallmatrix}0\\0\\0\\0\\0\end{smallmatrix} \\ Y & 0 & X & \begin{smallmatrix}1\\1\\1\\1\\1\end{smallmatrix} \\ I & X & X & \begin{smallmatrix}0\\0\\0\\0\\0\end{smallmatrix} \\ 00000 & 11111 & 00000 & 0 \end{array}\right]$$

A few other examples establishing small two color numbers can be cast in this same light.

Motivated by these examples and the difficulty of obtaining results by standard techniques, we decided to adopt a non-traditional approach to determining Ramsey numbers. Rather than attempting to construct a graph which would establish a given Ramsey number, we decided to reverse the procedure. We focused on a family of constructions and then sought to discover Ramsey numbers which those constructions would serve to establish. Naturally, we concentrated our attention on constructions similar to those which had proved successful in other cases, namely, those based on a C_5-decomposition of the adjacency matrix.

Before discussing our results, we establish some notation and terminology. We adopt that of Harary [10], except that we refer to vertices and edges rather than to points and lines, and use $e(G)$ to denote the number of edges and $|G|$ to denote the number of vertices of the graph G. Like Harary, we use $\mathrm{ex}(n, G)$ to denote the maximum number of edges in a graph on n vertices not containing a copy of G. We will denote by R, G and B, respectively, the red, green and blue subgraphs induced by a three-coloring of a complete graph. In that case, if v is a fixed vertex, we will denote the set of vertices adjacent to v in red, green and blue by R_v, G_v and B_v, respectively.

Main results

Our first theorem shows that $r(C_4, K_3, K_3)$ is 17, identical to the three-color number for K_3 [9]. This is somewhat surprising, since C_4-free graphs have far fewer edges than K_3-free graphs (see [8, p. 127] for a discussion of this topic).

To facilitate the proof, we note that in [4] it was shown that $r(K_3, C_4) = 7$. And, as the reader can verify, there are only two (K_3, C_4)-colorings of K_6. Hence we have:

Lemma 1. *$r(K_3, C_4) = 7$ and the only two (K_3, C_4)-colorings of K_6 are given by*:

(i) $R = 2K_3$

(ii) $\bar{R} = 2K_3 + $ *one edge.*

Theorem 1. $r(C_4, K_3, K_3) = r(K_3, K_3, K_3) = 17.$

Proof. To show $r(C_4, K_3, K_3) = 17$, we construct a coloring of K_{16}. The adjacency matrices, M_i, are described in terms of X and Y, defined above. For color 1, where C_4 is to be avoided, we have

$$M_1 = \begin{bmatrix} X & I & I & \begin{matrix}0\\0\\0\\0\\0\end{matrix} \\ I & Y & I & \begin{matrix}0\\0\\0\\0\\0\end{matrix} \\ I & I & 0 & \begin{matrix}1\\1\\1\\1\\1\end{matrix} \\ 00000 & 00000 & 11111 & 0 \end{bmatrix}$$

Note that vertices 1–10 induce a Petersen graph. For the other two colors, we have

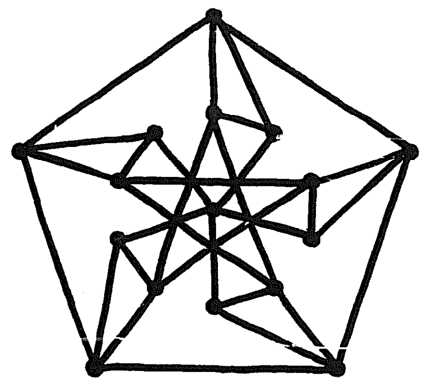
$$M_2 = \begin{bmatrix} 0 & X & X & \begin{matrix}1\\1\\1\\1\\1\end{matrix} \\ X & X & X & \begin{matrix}0\\0\\0\\0\\0\end{matrix} \\ X & X & X & \begin{matrix}0\\0\\0\\0\\0\end{matrix} \\ 11111 & 00000 & 00000 & 0 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} Y & Y & Y & \begin{matrix}0\\0\\0\\0\\0\end{matrix} \\ Y & 0 & Y & \begin{matrix}1\\1\\1\\1\\1\end{matrix} \\ Y & Y & Y & \begin{matrix}0\\0\\0\\0\\0\end{matrix} \\ 00000 & 11111 & 00000 & 0 \end{bmatrix}$$

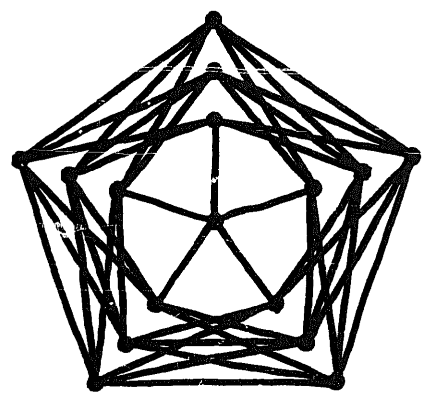
Since the graphs in colours 2 and 3 are isomorphic, only two distinct graphs are involved in this coloring. They are displayed in Fig. 1.

In order to show that $r(C_4, K_3, K_3) \leq 17$, assume that (R, G, B) is a (C_4, K_3, K_3)-coloring of K_{17}. We shall arrive at a contradition. From the lemma, we have $\delta(R) \geq 4$. Two cases are considered:

Case (*A*). In R there is a vertex v with red degree 4 such that $R_v = 2K_2$.



(a)



(b)

Fig. 1. The two distinct graphs used in the coloring showing $r(C_4, K_3, K_3) = 17$. Graph (a) is C_4-free, while graph (b) is K_3-free.

Again from the lemma, $|G_v| = |B_v| = 6$ and the red graphs induced by G_v and B_v each contain two disjoint red triangles. It is not hard to verify that there are at most 4 red edges joining G_v and B_v, and at most 8 red edges joining R_v and $(G_v \cup B_v)$. So the total number of red edges is at most $6 + 7 + 7 + 4 + 8 = 32$. This contradicts $\delta(R) \geqslant 4$.

Case (B). There is no vertex satisfying the conditions of (A).

Since $r(K_3, K_3, K_3) = 17$, there is a red K_3. Let x_1, x_2 and x_3 be the vertices of the K_3. Since (A) does not hold, for $1 \leqslant i \leqslant 3$, x_i has two neighbors, y_i and z_i, external to the K_3, which are independent in R. Since R does not contain C_4, these 6 vertices are distinct and form an independent set of size 6. From $r(K_3, K_3) = 6$, we conclude that the coloring contains a blue or green K_3, a contradiction. □

We attempted to determine the three-color Ramsey numbers for all possible combinations of C_4 and K_3. Since the values $r(K_3;\ 3) = 17$ and $r(C_4;\ 3) = 10$ [1] were known, Theorem 1 leaves only $r(C_4, C_4, K_3)$ to be determined. Obtaining an exact value here may be difficult. We are convinced that the correct answer is 12, but completing a proof of the upper bound is onerous. The difficulty seems to be that one can "almost" color K_{12} satisfactorily; we have colorings with exactly one forbidden subgraph (which may be either C_4 or K_3).

Our next coloring also relates to C_4. Here the matrices X and Y are again used to described the coloring.

Theorem 2. *If G is a connected graph on four or more vertices, then $r(C_4, C_4, C_4, G) \geq 16$.*

Proof. We construct a 4-coloring of K_{15} with no monochromatic C_4 in any of the first three colors and with no connected graph of order 4 in the fourth color. The adjacency matrices M_i, $i \leq 1 \leq 4$, for the four color graphs are given in terms of the 5×5 submatrices X and Y.

$$M_1 = \begin{bmatrix} X & 0 & Y \\ 0 & Y & X \\ Y & X & 0 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 0 & Y & X \\ Y & X & 0 \\ X & 0 & Y \end{bmatrix},$$

and

$$M_3 = \begin{bmatrix} Y & X & 0 \\ X & 0 & Y \\ 0 & Y & X \end{bmatrix}$$

The graphs for these three colors are isomorphic. A drawing of the graph is given in Fig. 2. By subtraction, we see that

$$M_4 = \begin{bmatrix} 0 & I & I \\ I & 0 & I \\ I & I & 0 \end{bmatrix}.$$

Since this is the adjacency matrix for $5K_3$, the theorem follows. $\square$

Corollary. *If T is a tree of 4 vertices (i.e., P_4 or $K_{1,3}$), then $r(C_4, C_4, C_4, T) = 16$.*

Proof. The lower bound follows from the theorem. The upper bound follows from the following three extremal results:

(a) $\mathrm{ex}(16, K_{1,3}) = 16$.
(b) $\mathrm{ex}(16, P_4) = 15$.
(c) $\mathrm{ex}(16, C_4) < 35$.

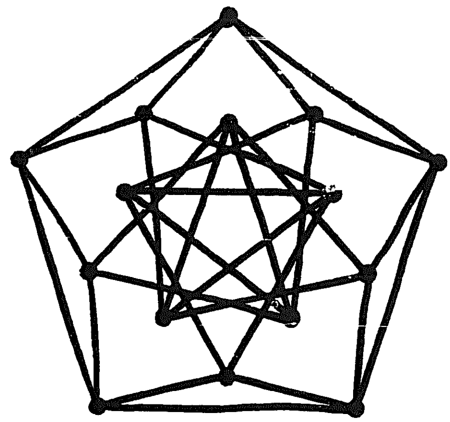


Fig. 2. The C_4-free graph used in Theorem 3.

Both (a) and (b) are easy to verify. To prove (c) suppose G has order 16, 35 edges and no C_4 subgraphs. Let $\{d_i\}$ be its degree sequence. Then since G has no C_4 subgraphs,

$$\binom{16}{2} 120 \leq \sum_{i=1}^{16} \binom{d_i}{2}.$$

But with 35 edges, we have

$$\sum_{i=1}^{16} d_i = 70.$$

This implies

$$\sum_{i=1}^{16} \binom{d_i}{2} \geq 120.$$

Hence equality holds and every two vertices of G have exactly one common neighbor. But by the Friendship Theorem [7] no such G can exist. □

Our next coloring based on C_5 decompositions is used to determine the two-color Ramsey number of two complete bipartite graphs. We note that for such graphs, diagonal numbers have been completely determined only for the stars [11]. For $K_{2,n}$, the problem is clearly related to that of books (see [13]), and determining exact values may depend on settling existence questions for strongly regular graphs. In view of this, $r(K_{2,3}, K_{2,4})$ seemed to be a natural first candidate for investigation.

Theorem 3. $r(K_{2,3}, K_{2,4}) = 12$.

Proof. The graph shown in Fig. 3 establishes the lower bound. Note that in this graph the decomposition is obtained by considering the outer pentagon and inner pentagram of the figure. To prove the upper bound, we assume the existence of a suitable coloring of K_{12} in colors red and green and arrive at a contradiction.

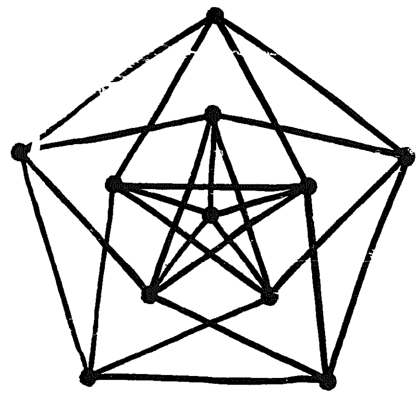


Fig. 3. A graph which shows $r(K_{2,3}, K_{2,4}) \geqslant 11$.

Let v be an arbitrary vertex, let $|R_v| = r$ and $|G_v| = g$, and let m represent the number of red edges joining R_v and G_v. Then we have $r(8-r) \leqslant m \leqslant 2g$. The inequality on the left follows from the observation that $u \in R_v$ has at most three green neighbors in G_v, since otherwise u and w have four common green neighbors. The inequality on the right is derived from the fact that any $w \in G_v$ has at most two red neighbors in R_v.

Since $g = 11 - r$, we have $r(8-r) \leqslant 2(11-r)$. Hence $r < 3$ or $r \geqslant 7$. But any two vertices in R have at most one common red neighbor in G_v, so $g \geqslant 2(8-r) - 1$, so $r \geqslant 4$. Thus $\delta(R) \geqslant 7$. To see that this is impossible, choose vertices v_1 and v_2 which are adjacent in green. They each have seven neighbors, but at most two common neighbors. This is impossible in a graph of order 12. Hence the assumed coloring does not exist and the theorem is proved. □

Conclusion

We conclude by proposing some possible directions which future investigations in this area might take.

One natural next step in this investigation would be to consider constructions based on an arbitrary Paley graph [2] rather than C_5. The recent examples given by Shearer in [14] to establish lower bounds for some classical Ramsey numbers can be viewed in this way. If we let W represent the adjacency matrix of an arbitrary Paley graph and let $Z = J - I - W$, then Shearer's constructions can be described the the adjacency matrix:

$$\begin{bmatrix} W & Z & \begin{matrix}0\\0\\0\\0\\0\end{matrix} & \begin{matrix}1\\1\\1\\1\\1\end{matrix} \\ Z & W & \begin{matrix}1\\1\\1\\1\\1\end{matrix} & \begin{matrix}0\\0\\0\\0\\0\end{matrix} \\ 00000 & 11111 & 0 & 1 \\ 11111 & 00000 & 1 & 0 \end{bmatrix}$$

Another direction which could be pursued is to attempt to devise related constructions which could be used to establish good asymptotic lower bounds for multicolor Ramsey numbers.

Acknowledgment

The authors wish to thank both referees: one for streamlining the proof of the upper bound in Theorem 1, the other for several comments concerning the presentation of our results.

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